

ON THE STABILIZATION OF EMBEDDED THICKENINGS

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ABSTRACT. We define a space of relative embedded thickenings of a map from a finite complex to a Poincaré Duality space, and show that there is a highly connected *stabilization map* induced by fiberwise suspension. As a result, we obtain a generalization to the Poincaré Duality category of a smooth stabilization theorem of Connolly and Williams.

CONTENTS

1. Introduction	1
2. The Space of Embedded Thickenings	5
3. The 4-D Face Theorem	12
4. Decompression and Section Data for Embedded Thickenings	16
5. Proof of the Stabilization Theorem	26
6. A Generalization of Smooth Stabilization	32
References	33

1. INTRODUCTION

A *smooth embedding up to homotopy* of a finite complex K in a manifold M is given by a pair (h, N) such that

- $N \subset M$ is a compact, codimension zero, smooth submanifold
- $h : K \xrightarrow{\sim} N$ is a homotopy equivalence

A natural question to ask is “When does a smooth embedding up to homotopy desuspend?”. Specifically, in the case that $M = S^n$, one can ask:

Question. When is a smooth embedding up to homotopy of ΣK in S^{n+1} induced by a smooth embedding up to homotopy of K in S^n ?

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This question was answered by Connolly and Williams in [CW78] in the case that K is 1-connected. In particular, call two smooth embeddings up to homotopy (h_0, N_0) and (h_1, N_1) of K in S^n *concordant* if

- There is an h -cobordism $W \subset S^n \times D^1$ between N_0 and N_1
- There is a homotopy equivalence $H : K \times D^1 \xrightarrow{\cong} W$ extending h_0 and h_1

Let $\pi_0(E_n(K))$ denote the set of such concordance classes.

Theorem [CW78, Stabilization Theorem]. *Assume K is a homotopy finite, r -connected ($r \geq 1$) complex of dimension $k \leq n - 3$, $n \geq 6$. Further, assume that $k - r \leq 2$ for $n \leq 7$. Then the natural suspension map*

$$\Sigma : \pi_0(E_n(K)) \rightarrow \pi_0(E_{n+1}(\Sigma K))$$

is surjective for $2(k - r) \leq n$ and injective for $2(k - r) \leq n - 1$.

The object of this paper is to generalize the result above to the Poincaré Duality category. In contrast to the surgery-theoretic methods employed by Connolly and Williams, we provide a manifold-free proof using fiberwise homotopy theory (cf Remark 1.2). Our generalization will actually follow from a “Freudenthal-like” stabilization theorem for our analogs of smooth embeddings up to homotopy, which we call *embedded thickenings*. Roughly, an embedded thickening is specified by a map $f : K \rightarrow X$ from a finite complex K to a Poincaré Duality space X , along with a pair of spaces (C, A) and a rule for gluing K and C together along A to form X , up to homotopy. In the event that X has a boundary ∂X , we define a *relative embedded thickening* for a map of pairs of spaces $f : (K, L) \rightarrow (X, \partial X)$ by specifying gluing data, similar to that given above, along with the assumption that an embedded thickening for the map $L \rightarrow \partial X$ is given. We defer the details to Definitions 2.2 and 2.3 below.

Main Results. We will write $\text{hodim}(K, L) = k$ if K can be obtained from L , up to homotopy, by attaching cells of dimension at most k . Let \mathcal{S}_X denote the (unreduced) fiberwise suspension functor (see Definition 2.9). In what follows, we define a moduli space

$$\mathbb{E}_f(K, X \text{ rel } L)$$

of embedded thickenings for a given map $f : (K, L) \rightarrow (X, \partial X)$ of (homotopy) finite pairs, along with a *stabilization map*

$$\sigma : \mathbb{E}_f(K, X \text{ rel } L) \rightarrow \mathbb{E}_{\mathcal{S}_X f}(\mathcal{S}_X K, X \times D^1 \text{ rel } \mathcal{S}_X L).$$

Our main result concerns the connectivity of the stabilization map:

Theorem A (Stabilization). *Let $f: (K, L) \rightarrow (X, \partial X)$ be a map from a cofibration pair of homotopy finite spaces (K, L) , with $\text{hodim}(K, L) = k$, to a Poincaré Duality pair $(X, \partial X)$ of dimension n . Assume that $f: K \rightarrow X$ is r -connected ($r \geq 1$) and that $k \leq n - 3$. Then the stabilization map*

$$\sigma: \mathbb{E}_f(K, X \text{ rel } L) \rightarrow \mathbb{E}_{\mathcal{S}_X f}(\mathcal{S}_X K, X \times D^1 \text{ rel } \mathcal{S}_X L)$$

is $(n - 2(k - r) - 3)$ -connected.

In the case that $X = S^n$ and $L = \emptyset$, there is an evident “collapse” map

$$\mathbb{E}_{\mathcal{S}_{S^n} f}(\mathcal{S}_{S^n} K, S^n \times D^1 \text{ rel } \mathcal{S}_{S^n} \emptyset) \xrightarrow{c} \mathbb{E}_{\mathcal{S} f}(\mathcal{S} K, S^{n+1})$$

where \mathcal{S} is the usual (unreduced) suspension functor. We can precompose this map with the stabilization map to get a map

$$\mathbb{E}_f(K, S^n) \xrightarrow{\text{co}\sigma} \mathbb{E}_{\mathcal{S} f}(\mathcal{S} K, S^{n+1}).$$

In what follows, we show that the map c above induces a surjection on π_0 and, as a consequence of Theorem A, obtain our generalization of the Stabilization Theorem of Connolly and Williams:

Theorem B. *Let K be a homotopy finite complex with $\text{hodim}(K) = k \leq n - 3$. Assume that $f: K \rightarrow S^n$ is an r -connected map of spaces, $r \geq 1$. Then the induced map*

$$\pi_0(c \circ \sigma): \pi_0(\mathbb{E}_f(K, S^n)) \rightarrow \pi_0(\mathbb{E}_{\mathcal{S} f}(\mathcal{S} K, S^{n+1}))$$

is surjective for $n \geq 2(k - r) + 3$ and injective for $n \geq 2(k - r) + 4$.

Remark 1.1. Theorem A serves as a classification tool for embedded thickenings for f . In a future paper [Pe], we will construct another space, $\mathbb{S}\mathbb{W}_f(K, X \text{ rel } L)$, which can be thought of as a kind of moduli space of unstable fiberwise duals of embedded thickenings with underlying map $f: (K, L) \rightarrow (X, \partial X)$. We will then establish the existence of a *classification map*

$$\theta: \mathbb{E}_f(K, X \text{ rel } L) \rightarrow \mathbb{S}\mathbb{W}_f(K, X \text{ rel } L)$$

and use Theorem A to prove that this map is highly-connected. This, in turn, will pave the way to an obstruction theory for the existence of embedded thickenings, as well as enumeration results and computations of the space $\mathbb{E}_f(K, X \text{ rel } L)$ in some special cases.

Remark 1.2. After a suitable application of the Browder-Casson-Sullivan-Wall theorem [Kl00, Theorem 5.3], our Theorem B implies the stabilization theorem of Connolly and Williams without the assumptions that $n \geq 6$ and $k - r \leq 2$ for $n \leq 7$. As mentioned above, in contrast to the surgery-theoretic methods used in [CW78], the proofs of our theorems are purely homotopy-theoretic. We rely heavily on *fiberwise homotopy theory*, a large portion of which can be found in [Kl99] and [Kl02a]. The proofs of our results also rely to a great extent on the “higher homotopy excision” theorems of Goodwillie [Go92].

Preliminaries.

Notation and Conventions. Let \mathcal{T} denote the category of compactly-generated topological spaces with the Quillen model structure [Qu67] based on weak homotopy equivalences and (Serre) fibrations. Constructions in \mathcal{T} , such as products and function spaces, will be understood to be topologized using the compactly-generated topology. In what follows, “space” will mean “cofibrant object of \mathcal{T} ” unless otherwise stated. A space X is n -connected if $\pi_i(X) = 0$ for every $i \leq n$ and for every choice of basepoint. In particular, a nonempty space is always (-1) -connected and, by convention, the empty space is (-2) -connected. A map of spaces $X \rightarrow Y$ is n -connected if its homotopy fiber, with respect to any basepoint of Y , is an $(n - 1)$ connected space. A weak equivalence in \mathcal{T} is an ∞ -connected map.

In the event that we specify a basepoint $*$ for a given space X , we will assume that the inclusion $*$ \rightarrow X is a cofibration. We will also write (\overline{X}, Y) to denote the mapping cylinder \overline{X} of a given a map $Y \rightarrow X$ with the inclusion of Y as $Y \times 0$.

We will use the language of model categories frequently. In particular, every object in a given model category has a (co)fibrant replacement. We will make such replacements, when necessary, often without changing notation. For a relevant discussion of model categories see [Ho99]. We will assume that the reader is familiar with the language of homotopy (co)limits and homotopy (co)cartesian diagrams. This language is developed in [Go92].

Lastly, a note about set theory. Most of the categories that we will work with are not small. To avoid set-theoretic difficulties when working with such categories, we fix a Grothendieck universe \mathcal{U} and use only \mathcal{U} -sets to form the objects of the category. In particular, we will write $|\mathcal{C}|$ for the geometric realization of the nerve of the (possibly large) category \mathcal{C} . This convention is not the only feasible option (see, eg, [GK08]).

Factorization Categories. Let $\mathcal{T}(A \xrightarrow{f} B)$ denote the category whose objects are triples (i, Y, j) such that Y is an object of \mathcal{T} , $i: A \rightarrow Y$ and $j: Y \rightarrow B$ are morphisms in \mathcal{T} , and $j \circ i = f$. A morphism $(i, Y, j) \rightarrow (i', Y', j')$ is a morphism $g: Y \rightarrow Y'$ in \mathcal{T} such that $g \circ i = i'$ and $j' \circ g = j$. It is well-known that $\mathcal{T}(A \xrightarrow{f} B)$ is a model category whose weak equivalences and (co)fibrations are determined by the forgetful functor to \mathcal{T} . In particular, an object (i, Y, j) of $\mathcal{T}(A \xrightarrow{f} B)$ is fibrant if $j: Y \rightarrow B$ is a fibration in \mathcal{T} , and cofibrant if $i: A \rightarrow Y$ is a cofibration in \mathcal{T} . Note that $\mathcal{T}(\emptyset \rightarrow X)$ is the category $(\mathcal{T} \downarrow X)$ of spaces over X . An object Y of $(\mathcal{T} \downarrow X)$ is called m -connected if the structure map $Y \rightarrow X$ is an $(m + 1)$ -connected map of spaces.

Outline. In Section 2, we recall the notions of Poincaré Duality spaces and embedded thickenings. We then construct the space of embedded thickenings for a given map f , along with the stabilization map. Section 3 is devoted to stating and proving a technical theorem which we call the “4-Dimensional Face Theorem”. This theorem is a generalization of the “Face Theorem” of Klein [Kl99, Theorem 5.1] and concerns the degree to which a 2-dimensional face of a given 4-dimensional cubical diagram is cocartesian. In Section 4 we recall the notion of *decompressing* an embedded thickening and use fiberwise homotopy theory to construct sections to structure maps in given embedded thickenings. In Section 5, we use the maps constructed in Section 4, along with the 4-D Face Theorem, to prove Theorem A. In Section 6 we prove Theorem B.

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2. THE SPACE OF EMBEDDED THICKENINGS

The following definitions can be found in [Kl99] and [Kl02b], and are included for the sake of completeness.

Definition 2.1. Let X be a homotopy finite space equipped with a local coefficient system \mathcal{L} which is pointwise free abelian of rank one. Let $[X]$ denote a homology class in $H_n(X; \mathcal{L})$. The data $(X, \mathcal{L}, [X])$ equip X with the structure of a *Poincaré Duality space of formal dimension n* if cap product with $[X]$ induces an isomorphism

$$\cap[X]: H^*(X, \mathcal{L}) \xrightarrow{\cong} H_{n-*}(X; \mathcal{L} \otimes \mathcal{M})$$

for every local coefficient system \mathcal{M} . We call $[X]$ the *fundamental class* of X and we will refer to such a space X as a *PD space of dimension n* . A cofibration pair $(X, \partial X)$ of homotopy finite spaces along with \mathcal{L} and a class $[X] \in H_n(X, \partial X; \mathcal{L})$ is called a *Poincaré Duality pair of formal dimension n* if

- For all local systems \mathcal{M} , there is an induced isomorphism

$$\cap[X]: H^*(X; \mathcal{L}) \xrightarrow{\cong} H_{n-*}(X, \partial X; \mathcal{L} \otimes \mathcal{M})$$

- The restriction of \mathcal{L} to ∂X along with the image of the fundamental class $[X]$ under the boundary homomorphism $H_n(X, \partial X; \mathcal{L}) \rightarrow H_{n-1}(\partial X; \mathcal{L})$ equips ∂X with the structure of a PD space of dimension $n - 1$.

We will call such a pair $(X, \partial X)$ a *PD pair of dimension n* .

Definition 2.2. Let $K \xrightarrow{f} X$ denote a map from a connected, homotopy finite space K to a PD space X or PD pair $(X, \partial X)$ of dimension n . An *embedded thickening*¹ of f is specified by homotopy finite spaces A and C along with a choice of factorization $\partial X \rightarrow C \rightarrow X$ fitting into a commutative diagram

$$(\mathcal{D}) \quad \begin{array}{ccc} A & \longrightarrow & C \longleftarrow \partial X \\ \downarrow & & \downarrow \\ K & \xrightarrow{f} & X \end{array}$$

such that

- (*Stratification*) The square is ∞ -cocartesian, ie, there is a weak homotopy equivalence of spaces $K \cup_A C \simeq X$.
- (*Poincaré Duality*) The image of the fundamental class $[X]$ under the composite

$$H_n(X, \partial X) \cong H_n(\overline{X}, \partial X) \rightarrow H_n(\overline{X}, C) \cong H_n(\overline{K}, A)$$

equips (\overline{K}, A) with the structure of a PD pair and, similarly, the image of $[X]$ with respect to the map $H_n(X, \partial X) \rightarrow H_n(\overline{C}, \partial X \amalg A)$ equips $(\overline{C}, \partial X \amalg A)$ with the structure of a PD pair.

¹We use the term “embedded thickening” to conform to the terminology set forth in [Kl02a]. Such embeddings were formerly called *PD Embeddings* in [Kl99].

- (iii) (*Weak Transversality*) If $\text{hodim}(K) \leq k$, then the map $A \rightarrow K$ is $(n - k - 1)$ -connected.

We call A the *gluing space*, C the *complement*, and f the *underlying map* of the embedded thickening \mathcal{D} .

We can relativize the above as follows. Let (K, L) be a cofibration pair, with K and L homotopy finite, and let $(X, \partial X)$ be a PD pair of dimension n . Recall that $\text{hodim}(K, L) = k$ if K can be obtained from L , up to homotopy, by attaching cells of at most dimension k . Fix a map $f = (f_K, f_L): (K, L) \rightarrow (X, \partial X)$.

Definition 2.3. A *relative embedded thickening of f* consists of a commutative diagram of pairs of homotopy finite spaces

$$(\mathcal{E}) \quad \begin{array}{ccc} (A_K, A_L) & \longrightarrow & (C_K, C_L) \\ \downarrow & & \downarrow \\ (K, L) & \xrightarrow{f} & (X, \partial X) \end{array}$$

such that

- (i) (*Stratification*) Each of the associated squares of spaces

$$(\mathcal{D}_K) \quad \begin{array}{ccc} A_K & \longrightarrow & C_K \\ \downarrow & & \downarrow \\ K & \xrightarrow{f_K} & X \end{array} \quad (\mathcal{D}_L) \quad \begin{array}{ccc} A_L & \longrightarrow & C_L \\ \downarrow & & \downarrow \\ L & \xrightarrow{f_L} & \partial X \end{array}$$

is ∞ -cocartesian, and the square \mathcal{D}_L is an embedded thickening for f_L .

- (ii) (*Poincaré Duality*) The image of the fundamental class $[X]$ under the composite

$$H_n(X, \partial X) \cong H_n(\overline{X}, \partial X) \rightarrow H_n(\overline{X}, \partial X \cup_{C_L} C_K) \cong H_n(\overline{K}, L \cup_{A_L} A_K)$$

equips $(\overline{K}, L \cup_{A_L} A_K)$ with the structure of a PD pair and, similarly, the image of $[X]$ with respect to the map

$$H_n(X, \partial X) \rightarrow H_n(\overline{C_K}, C_L \cup_{A_L} A_K)$$

equips $(\overline{C_K}, C_L \cup_{A_L} A_K)$ with the structure of a PD pair. (Here, coefficients are given by pulling back the given local system on X).

- (iii) (*Weak Transversality*) If $\text{hodim}(K, L) \leq k$, then the map $A_K \rightarrow K$ is $(n - k - 1)$ -connected.

Remark 2.4. In the situation above, if an embedded thickening \mathcal{D}_L of f_L is given, then an embedded thickening of f which coincides with \mathcal{D}_L on L is said to be an *embedded thickening of f relative to \mathcal{D}_L* . By taking $L = \emptyset$, we recover the definition of embedded thickening given in 2.2 above.

Definition 2.5. Let \mathcal{D}_0 and \mathcal{D}_1 be embedded thickenings with underlying maps $f_0, f_1: K \rightarrow X$ and suppose that we are given a homotopy $F: K \times D^1 \rightarrow X$ from f_0 to f_1 . Then we have an associated embedded thickening with underlying map

$$f_0 + f_1: K \amalg K \rightarrow \partial(X \times D^1).$$

Denote this embedding by $\mathcal{D}_{K \amalg K}$ and consider the associated map of pairs

$$F: (K \times D^1, K \amalg K) \rightarrow (X \times D^1, \partial(X \times D^1)).$$

A *concordance* from \mathcal{D}_0 to \mathcal{D}_1 is an embedded thickening of F relative to $\mathcal{D}_{K \amalg K}$.

The Space $\mathbb{E}_f(K, X \text{ rel } L)$. Fix an object $(f, K) \in (\mathcal{T} \downarrow X)$ and define a category $\mathcal{D}_f(K, X)$ as follows. An object of $\mathcal{D}_f(K, X)$ is a commutative square of spaces

$$(\mathcal{D}) \quad \begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ K & \xrightarrow{f} & X \end{array}$$

and a morphism in $\mathcal{D}_f(K, X)$ is a map of pairs of spaces $(C, A) \rightarrow (C', A')$ that covers f . That is, a morphism $\mathcal{D} \rightarrow \mathcal{D}'$ is given by a commutative diagram

$$(\mathcal{D} \rightarrow \mathcal{D}') \quad \begin{array}{ccccc} & & A & \longrightarrow & C \\ & \swarrow & \downarrow & & \swarrow \\ A' & \longrightarrow & C' & & \\ \downarrow & & \downarrow & & \downarrow \\ & & K & \longrightarrow & X \\ \downarrow & \parallel & \downarrow & \parallel & \\ K & \xrightarrow{f} & X & & \end{array}$$

Proposition 2.6. $\mathcal{D}_f(K, X)$ is a model category.

Proof. Let \mathcal{T}^2 denote the arrow category of \mathcal{T} , equipped with the projective model structure. The fixed object $(f, K) \in (\mathcal{T} \downarrow X)$ used to define $\mathcal{D}_f(K, X)$ determines an object $K \xrightarrow{f} X$ of \mathcal{T}^2 . The proposition then follows by noting that the category $\mathcal{D}_f(K, X)$ is the same as the model category $(\mathcal{T}^2 \downarrow f)$. \square

It follows (eg, from [Ho99, Theorem 5.1.3]) that a morphism $(C, A) \rightarrow (C', A')$ in $\mathcal{D}_f(K, X)$ is a

- *weak equivalence* (respectively, *fibration*) if each of the maps $A \rightarrow A'$ and $C \rightarrow C'$ are weak equivalences (respectively, fibrations) in \mathcal{T}
- *cofibration* if both $A \rightarrow A'$ and the induced map $A' \cup_A C \rightarrow C'$ are cofibrations in \mathcal{T} .

In particular, the object \mathcal{D} of $\mathcal{D}_f(K, X)$ is cofibrant (respectively, fibrant) precisely when A is cofibrant in \mathcal{T} and the map $A \rightarrow C$ is a cofibration in \mathcal{T} (respectively, when both of the maps $A \rightarrow K$ and $C \rightarrow X$ are fibrations in \mathcal{T}).

Remark 2.7. In the situation of Definition 2.3, a more general version of the model structure above exists. In this case, we have a category $\mathcal{D}_f(K, X \text{ rel } L)$ in which the objects are commutative squares

$$(\mathcal{E}) \quad \begin{array}{ccc} (A_K, A_L) & \longrightarrow & (C_K, C_L) \\ \downarrow & & \downarrow \\ (K, L) & \xrightarrow{f} & (X, \partial X) \end{array}$$

as in Definition 2.3. A morphism is a square of pairs

$$\begin{array}{ccc} (A_K, A_L) & \longrightarrow & (C_K, C_L) \\ \downarrow & & \downarrow \\ (A'_K, A'_L) & \longrightarrow & (C'_K, C'_L) \end{array}$$

covering the map $(K, L) \xrightarrow{f} (X, \partial X)$ in \mathcal{E} . The weak equivalences in $\mathcal{D}_f(K, X \text{ rel } L)$ are morphisms as above in which both of the vertical maps are weak homotopy equivalences of pairs. The object \mathcal{E} is fibrant if the vertical maps in \mathcal{E} are both fibrations between the larger spaces that restrict to fibrations of the subspaces. Moreover, \mathcal{E} is cofibrant when A_L is cofibrant in \mathcal{T} , (A_K, A_L) is a cofibration pair of spaces, and

the induced map $A_K \cup_{A_L} C_L \rightarrow C_K$ is a cofibration in \mathcal{T} . In the case that L and A_L are both empty, and $C_L = \partial X$, we recover the model structure on $D_f(K, X)$ described above.

We can now define the space of embedded thickenings of a given map of homotopy finite, cofibration pairs $(K, L) \xrightarrow{f} (X, \partial X)$ such that K is connected and $(X, \partial X)$ is a PD pair of dimension n . To this end, for such a map f , let $w\mathcal{D}_f(K, X \text{ rel } L)$ denote the category with the same objects as $\mathcal{D}_f(K, X \text{ rel } L)$ but whose only morphisms are weak equivalences, and let $E_f(K, X \text{ rel } L)$ denote the full subcategory of $w\mathcal{D}_f(K, X \text{ rel } L)$ with objects given by commutative squares \mathcal{E} that determine embedded thickenings of f in the sense of Definition 2.3.

Definition 2.8. The *space of embedded thickenings of f* , denoted by $\mathbb{E}_f(K, X \text{ rel } L)$, is the geometric realization of the nerve of $E_f(K, X \text{ rel } L)$. That is,

$$\mathbb{E}_f(K, X \text{ rel } L) = |E_f(K, X \text{ rel } L)|.$$

The Stabilization Map. In this section we define the *stabilization map*

$$\sigma: \mathbb{E}_f(K, X \text{ rel } L) \rightarrow \mathbb{E}_{\mathcal{S}_X f}(\mathcal{S}_X K, X \times D^1 \text{ rel } \mathcal{S}_X L).$$

First we recall the notion of *fiberwise suspension*:

Definition 2.9. Given an object $(f, K) \in (\mathcal{T} \downarrow X)$, the *unreduced fiberwise suspension of K over X* is the double mapping cylinder

$$\mathcal{S}_X K = X \times 0 \cup_{f \times 0} K \times D^1 \cup_{f \times 1} X \times 1.$$

Remark 2.10. (cf [K199]) We may regard *fiberwise suspension over X* as a functor

$$\mathcal{S}_X: (\mathcal{T} \downarrow X) \rightarrow \mathcal{T}(X \amalg X \xrightarrow{\nabla} X)$$

where $\nabla: X \amalg X \rightarrow X$ is the fold map. It is straightforward to check that \mathcal{S}_X maps cofibrant objects to cofibrant objects. Note that $\mathcal{S}_X X = X \times D^1$. We will use both notations in what follows.

Let

$$(\mathcal{D}) \quad \begin{array}{ccc} (A_K, A_L) & \longrightarrow & (C_K, C_L) \\ \downarrow & & \downarrow \\ (K, L) & \xrightarrow{f=(f_K, f_L)} & (X, \partial X) \end{array}$$

be an object of $E_f(K, X \text{ rel } L)$. That is, \mathcal{D} is an embedded thickening for $f = (f_K, f_L)$ relative to the embedded thickening

$$(\mathcal{D}_L) \quad \begin{array}{ccc} A_L & \longrightarrow & C_L \\ \downarrow & & \downarrow \\ L & \xrightarrow{f_L} & \partial X \end{array}$$

We can picture the given Poincaré stratification of the pair $(X, \partial X)$ as follows:

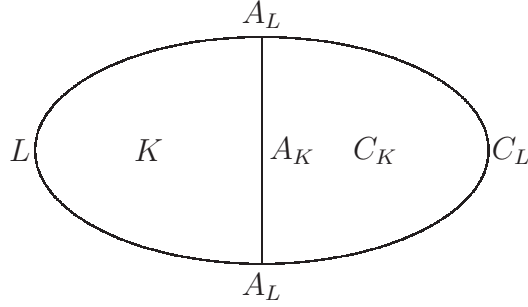


FIGURE 1. A PD Decomposition of $(X, \partial X)$

The motivation for the definition of the following functor comes from “crossing the stratification above with the interval”. Define a functor

$$\tilde{\sigma}: E_f(K, X \text{ rel } L) \rightarrow E_{\mathcal{S}_X f}(\mathcal{S}_X K, X \times D^1 \text{ rel } \mathcal{S}_X L)$$

on objects by

$$\begin{array}{ccc} (A_K, A_L) \longrightarrow (C_K, C_L) & & (\mathcal{S}_{C_K} A_K, \mathcal{S}_{C_L} A_L) \longrightarrow (C_K \times D^1, C_L \times D^1) \\ \downarrow & & \downarrow \\ (K, L) \xrightarrow{f} (X, \partial X) & \mapsto & (\mathcal{S}_X K, \mathcal{S}_X L) \xrightarrow{\mathcal{S}_X f} (X \times D^1, \partial(X \times D^1)) \\ (\mathcal{D}) & & (\tilde{\sigma} \mathcal{D}) \end{array}$$

Definition 2.11. The *stabilization map* σ is defined by applying the geometric realization functor to (the map of nerves induced by) $\tilde{\sigma}$:

$$\sigma = |\tilde{\sigma}|: \mathbb{E}_f(K, X \text{ rel } L) \rightarrow \mathbb{E}_{\mathcal{S}_X f}(\mathcal{S}_X K, X \times D^1 \text{ rel } \mathcal{S}_X L).$$

Establishing the connectivity of this map is the content of Theorem A.

3. THE 4-D FACE THEOREM

Let \mathbf{n} denote the category associated to the ordinal n as a poset, and write $P(\mathbf{n})$ for its poset of subsets, which we also regard as a category.

Theorem 3.1. (*4-Dimensional Face Theorem*)

Let $X: P(\mathbf{4}) \rightarrow \mathcal{T}$ be the 4-dimensional cubical diagram of spaces represented by the commutative diagram

$$\begin{array}{ccccc}
 & X_\emptyset & \longrightarrow & X_3 & \\
 & \swarrow & & \swarrow & \\
 X_1 & \longrightarrow & X_{13} & & \\
 \downarrow & & \downarrow & & \downarrow \\
 & X_2 & \longrightarrow & X_{23} & \\
 \downarrow & & \downarrow & & \downarrow \\
 X_{12} & \longrightarrow & X_{123} & \longrightarrow & X_{124} \longrightarrow X_{1234} \\
 & & & & \downarrow \\
 & & & & X_{24} \longrightarrow X_{234} \\
 & & & & \downarrow \\
 & & & & X_{134} \longrightarrow X_{34}
 \end{array}$$

Assume that

- The 4-cube X is ∞ -cartesian (ie, for $S \neq \emptyset$, the map $X_\emptyset \rightarrow \text{holim}_{S \subset P(\mathbf{4})} X(S)$ is a weak equivalence)
- The spaces X_S are connected for each nonempty $S \subset \mathbf{4}$
- Each 3-dimensional face which meets X_{1234} is strongly cocartesian
- Each map $X_S \rightarrow X_{S \cup \{i\}}$ is k_i -connected for S and $\{i\}$ nonempty subsets of $\mathbf{4}$, $i \notin S$.
- $k_i, k_j \geq 2$ for some $i \neq j$.

Then each of the squares

$$\begin{array}{ccc}
 X_\emptyset & \longrightarrow & X_j \\
 \downarrow & & \downarrow \\
 X_i & \longrightarrow & X_{ij}
 \end{array}$$

is $(\sum_{i=1}^4 k_i - 1)$ -cocartesian for $1 \leq i < j \leq 4$.

Remark 3.2. Let T be a nonempty subset of $\mathbf{4}$ and let $\partial_{\mathbf{4}-T} X$ denote the $|T|$ -face of X terminating in X_{1234} . Suppose each of these $|T|$ -faces is k_T -cartesian. One can easily check that $\min \{\sum_\alpha k_{T_\alpha}\} = \sum_{i=1}^4 k_i - 2$, where the minimum is taken over all partitions $\{T_\alpha\}_\alpha$ of $\mathbf{4}$ by nonempty subsets. By the dual Blakers-Massey Theorem [Go92, Theorem 2.6] X is $(\sum_{i=1}^4 k_i + 1)$ -cocartesian. Write X as a map of 3-cubes $Y \rightarrow Z$. By hypothesis, Z is strongly cocartesian. Let $H_*(X)$ denote the reduced

homology of the total cofiber of X , and similarly for Y and Z . Then $H_n(Z) = 0$ for all n and $H_n(X) = 0$ for $n \leq \sum_{i=1}^4 k_i + 1$. From the long exact sequence

$$\cdots \rightarrow H_n(Z) \rightarrow H_n(X) \rightarrow H_{n-1}(Y) \rightarrow H_{n-1}(Z) \rightarrow \cdots$$

we conclude that $H_n(Y) = 0$ for $n \leq \sum_{i=1}^4 k_i$. This result motivates the first claim made in the proof below.

Proof of 4D Face Theorem. As above, let X denote the 4-cube. Without loss in generality, assume that all of the maps in X are fibrations. Using Remark 3.2 and an argument similar to that given in the proof of [KL99, Theorem 5.1], our final hypothesis in the statement of the theorem guarantees that X_\emptyset is nonempty and connected.

Claim. Each of the 3-dimensional subcubical diagrams

$$\begin{array}{ccccc} & & X_\emptyset & \longrightarrow & X_k \\ & \swarrow & \downarrow & \swarrow & \downarrow \\ X_i & \longrightarrow & X_{ik} & & \\ \downarrow & & \downarrow & & \downarrow \\ & & X_j & \longrightarrow & X_{jk} \\ & \swarrow & \downarrow & \swarrow & \downarrow \\ X_{ij} & \longrightarrow & X_{ijk} & & \end{array}$$

is $\sum_{i=1}^4 k_i$ -cocartesian for $1 \leq i < j < k \leq 4$.

Proof of Claim: Choose one of the 3-cubes meeting X_\emptyset and call it Y , say

$$(Y) \quad \begin{array}{ccccc} & & X_\emptyset & \longrightarrow & X_3 \\ & \swarrow & \downarrow & \swarrow & \downarrow \\ X_1 & \longrightarrow & X_{13} & & \\ \downarrow & & \downarrow & & \downarrow \\ & & X_2 & \longrightarrow & X_{23} \\ & \swarrow & \downarrow & \swarrow & \downarrow \\ X_{12} & \longrightarrow & X_{123} & & \end{array}$$

It will be enough to prove the claim for this 3-cube. Let Z denote the 3-cube opposite Y in X . That is, Z is the 3-cube

$$(Z) \quad \begin{array}{ccccc} & & X_4 & \longrightarrow & X_{34} \\ & \swarrow & \downarrow & \swarrow & \downarrow \\ X_{14} & \longrightarrow & X_{134} & & \\ \downarrow & & \downarrow & & \downarrow \\ & & X_{24} & \longrightarrow & X_{234} \\ \swarrow & & \swarrow & & \swarrow \\ X_{124} & \longrightarrow & X_{1234} & & \end{array}$$

Let $\mathrm{hocolim}(X - X_{1234})$ denote the homotopy colimit of the restriction of X to the subposet of proper subsets of $P(\mathbf{4})$. Let $\mathrm{hocolim}(Y - X_{123})$ and $\mathrm{hocolim}(Z - X_{1234})$ denote the analogous homotopy colimits associated with Y and Z , respectively. The homotopy colimit of the diagram

$$X_{123} \leftarrow \mathrm{hocolim}(Y - X_{123}) \rightarrow \mathrm{hocolim}(Z - X_{1234}).$$

is equivalent to $\mathrm{hocolim}(X - X_{1234})$ and, thus, we have a commutative diagram

$$(2.1) \quad \begin{array}{ccc} \mathrm{hocolim}(Y - X_{123}) & \longrightarrow & \mathrm{hocolim}(Z - X_{1234}) \\ \downarrow & & \downarrow f \\ X_{123} & \longrightarrow & \mathrm{hocolim}(X - X_{1234}) \\ & \searrow & \downarrow g \\ & & X_{1234} \end{array} \quad \begin{array}{c} \sim \\ \curvearrowright \end{array}$$

where the square is ∞ -cocartesian. The equivalence labeled above arises from the assumption that Z is strongly cocartesian. The canonical map g is $(\sum_{i=1}^4 k_i + 1)$ -connected since X is $(\sum_{i=1}^4 k_i + 1)$ -cocartesian. By [Go92, Proposition 1.5(ii)] the map f is $(\sum_{i=1}^4 k_i)$ -connected.

Our goal is to show that the left vertical map in (2.1) is $(\sum_{i=1}^4 k_i)$ -connected. By [K199, Lemma 5.6(2)] it will be enough to show that the top horizontal map in (2.1) is 2-connected. To this end, note that each of the spaces in (2.1) admits a map to X_{1234} . Focusing only on the square above, the homotopy fibers of these maps (with respect to any choice of basepoint in X_{1234}) give rise to an ∞ -cocartesian square

$$(2.2) \quad \begin{array}{ccc} \mathrm{hofiber}(\mathrm{hocolim}(Y - X_{123}) \rightarrow X_{1234}) & \xrightarrow{\quad} & * \\ \downarrow & & \downarrow \\ \mathrm{hofiber}(X_{123} \rightarrow X_{1234}) & \twoheadrightarrow & \mathrm{hofiber}(\mathrm{hocolim}(X - X_{1234}) \rightarrow X_{1234}) \end{array}$$

Let s denote the connectivity of the space in the upper left corner of (2.2) and consider the diagram

$$\mathrm{hocolim}(Y - X_{123}) \rightarrow X_{1234} \xleftarrow{\sim} \mathrm{hocolim}(Z - X_{1234}).$$

We wish to show that $s \geq 1$, for then the first map above will be 2-connected and the claim will follow. To this end, note that the top horizontal map in (2.2) is $(s+1)$ -connected, so its homotopy cofiber is also $(s+1)$ -connected. Let C_{top} denote this homotopy cofiber. By hypothesis, the map $X_{123} \rightarrow X_{1234}$ is k_4 -connected. Since the map g in (2.1) is $(\sum_{i=1}^4 k_i + 1)$ -connected, it follows from ([Go92, Proposition 1.5(ii)]) that the bottom horizontal map in (2.1)

$$X_{123} \rightarrow \mathrm{hocolim}(X - X_{1234})$$

is k_4 -connected. Use this map to form the square

$$\begin{array}{ccc} X_{123} & \xrightarrow{\quad} & X_{1234} \\ \downarrow & & \downarrow = \\ \mathrm{hocolim}(X - X_{1234}) & \xrightarrow{\quad} & X_{1234} \end{array}$$

The induced map of horizontal homotopy fibers coming from this square is then k_4 -connected. But this induced map is precisely the bottom horizontal map in (2.2). Thus the homotopy cofiber of the bottom horizontal arrow in (2.2) is k_4 -connected. Let C_{bottom} denote this cofiber. Since (2.2) is ∞ -cocartesian, we have a weak equivalence $C_{top} \xrightarrow{\sim} C_{bottom}$. This implies that $s+1 = k_4$. But we assumed (without loss in generality) that $k_4 \geq 2$. Thus, $s \geq 1$.

Now we prove the statement concerning the degree to which each 2-face meeting X_\emptyset is cocartesian. Choose one of these 2-faces and call it V , say

$$(V) \quad \begin{array}{ccc} X_\emptyset & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & X_{12} \end{array}$$

It will be enough to show that V is $(\sum_{i=1}^4 k_i - 1)$ -cocartesian. By hypothesis, the 3-cube

$$\begin{array}{ccccc}
 & & X_3 & \longrightarrow & X_{34} \\
 & \swarrow & \downarrow & \swarrow & \downarrow \\
 X_{13} & \longrightarrow & X_{134} & & \\
 \downarrow & & \downarrow & & \downarrow \\
 & & X_{23} & \longrightarrow & X_{234} \\
 \downarrow & \swarrow & \downarrow & \swarrow & \downarrow \\
 X_{123} & \longrightarrow & X_{1234} & &
 \end{array}$$

is strongly cocartesian. Thus, the face

$$(W) \quad \begin{array}{ccc} X_3 & \longrightarrow & X_{13} \\ \downarrow & & \downarrow \\ X_{23} & \longrightarrow & X_{123} \end{array}$$

is ∞ -cocartesian. As in the claim above, there is a commutative diagram

$$(2.3) \quad \begin{array}{ccc} \mathrm{hocolim}(V - X_{12}) & \longrightarrow & \mathrm{hocolim}(W - X_{123}) \\ \downarrow & & \downarrow f' \\ X_{12} & \longrightarrow & \mathrm{hocolim}(Y - X_{123}) \\ & \searrow g' & \downarrow \\ & & X_{123} \end{array}$$

\sim
 \nearrow

Our goal is to show that the left vertical map in (2.3) is $(\sum_{i=1}^4 k_i - 1)$ -connected. This follows from an argument almost identical to the one just given. We omit the details. \square

4. DECOMPRESSION AND SECTION DATA FOR EMBEDDED THICKENINGS

Decompression. Let

$$(\mathcal{D}) \quad \begin{array}{ccc} (A_K, A_L) & \longrightarrow & (C_K, C_L) \\ \downarrow & & \downarrow \\ (K, L) & \xrightarrow{f} & (X, \partial X) \end{array}$$

denote an object of the category $E_f(K, X \text{ rel } L)$, which we will also think of as the corresponding 0-cell of the space $\mathbb{E}_f(K, X \text{ rel } L)$. Let

f^j denote the effect of the map $f: K \rightarrow X$ followed by the inclusion $X \rightarrow X \times D^j$. Define a functor (called the *decompression* functor²)

$$\tilde{\delta}: E_f(K, X \text{ rel } L) \rightarrow E_{f^1}(K, X \times D^1 \text{ rel } L)$$

on objects by sending \mathcal{D} to the relative embedded thickening

$$(\tilde{\delta}\mathcal{D}) \quad \begin{array}{ccc} (\mathcal{S}_K A_K, \mathcal{S}_L A_L) & \longrightarrow & (\mathcal{S}_X C_K, \mathcal{S}_X C_L) \\ \downarrow & & \downarrow \\ (K, L) & \xrightarrow{f^1} & (X \times D^1, \partial(X \times D^1)) \end{array}$$

Applying the geometric realization functor to (the map of nerves induced by) $\tilde{\delta}$ gives the *decompression map*

$$\delta = |\tilde{\delta}|: \mathbb{E}_f(K, X \text{ rel } L) \rightarrow \mathbb{E}_{f^1}(K, X \times D^1 \text{ rel } L).$$

Together, the stabilization and decompression maps give rise to the following square, which commutes up to preferred weak equivalence:

$$(\mathcal{D}_{\sigma, \delta}) \quad \begin{array}{ccc} \mathbb{E}_f(K, X \text{ rel } L) & \longrightarrow & \mathbb{E}_{\mathcal{S}_X f}(\mathcal{S}_X K, X \times D^1 \text{ rel } \mathcal{S}_X L) \\ \downarrow & & \downarrow \\ \mathbb{E}_{f^1}(K, X \times D^1 \text{ rel } L) & \longrightarrow & \mathbb{E}_{\mathcal{S}_X f^1}(\mathcal{S}_X K, X \times D^2 \text{ rel } \mathcal{S}_X L) \end{array}$$

Remark 4.1. The bulk of the proof of Theorem A lies in proving that the square $\mathcal{D}_{\sigma, \delta}$ is 0-cartesian. The rest of this section will be devoted to constructing certain maps (sections to structure maps in given embedded thickenings) that will allow us to use our 4-D Face Theorem to prove this claim.

Lemma 4.2. *Let $g: X \rightarrow Y$ be a t -connected map of based spaces, and assume that $\text{conn}(X) = \text{conn}(Y) = s$. Then the square*

$$\begin{array}{ccc} X & \longrightarrow & \Omega \Sigma X \\ g \downarrow & & \downarrow \\ Y & \longrightarrow & \Omega \Sigma Y \end{array}$$

is $(s+t)$ -cartesian.

²For a detailed construction of the decompression functor, see [Kl02b, Definition 2.4].

Proof. Recall that for a based space Z with basepoint $*$, the homotopy fiber of the inclusion $*$ \hookrightarrow Z is the based loop space ΩZ . The square in question is then determined by taking the homotopy fibers of the horizontal maps in the ∞ -cocartesian 3-cube

$$(X) \quad \begin{array}{ccccc} & & X & \longrightarrow & * \\ & \swarrow & \downarrow & \swarrow & \downarrow \\ * & \xrightarrow{\quad} & \Sigma X & & * \\ & \downarrow g & \downarrow & & \downarrow \\ & & Y & \longrightarrow & * \\ & \swarrow & \downarrow & \swarrow & \downarrow \\ * & \xrightarrow{\quad} & \Sigma Y & & * \end{array}$$

For T a nonempty subset of $\{1, 2, 3\}$, let $k(T)$ denote the degree to which the front face $\partial^T X = \{V \rightarrow X(V) : V \subset T\}$ is cartesian. Labeling the “ $*$ ”s in the top face of X by 1 and 3, and labeling Y by 2, one can easily check that

$$k(\{1\}) = k(\{3\}) = s + 1, \quad k(\{1, 2\}) = k(\{2, 3\}) = t + 1, \quad k(\{2\}) = t$$

and $k(\{1, 3\}) = k(\{1, 2, 3\}) = \infty$

By the generalized Blakers-Massey Theorem [Go92, Theorem 2.5], X is $(1 - 3 + s + t + 1 + 1) = (s + t)$ -cartesian. An application of [Go92, Proposition 1.18] completes the proof. \square

Remark 4.3. (cf [Kl98, Page 320]) The fiberwise suspension \mathcal{S}_X admits a right adjoint

$$\mathcal{O}_X : \mathcal{T}(X \amalg X \xrightarrow{\nabla} X) \rightarrow (\mathcal{T} \downarrow X)$$

given on objects by

$$Y \mapsto \operatorname{holim}(X \xrightarrow{i_+} Y \xleftarrow{i_-} X)$$

where i_{\pm} denote the restrictions of $X \amalg X \rightarrow Y$ to each summand.

Lemma 4.4. *Let $Y \in (\mathcal{T} \downarrow X)$ be a cofibrant object which is m -connected. Then there is a morphism*

$$Y \rightarrow \mathcal{O}_X \mathcal{S}_X Y$$

of $(\mathcal{T} \downarrow X)$ which is $(2m + 1)$ -connected.

Proof. Let $g: Y \rightarrow X$ be the structure map associated with the object Y . Then by definition, we have the following ∞ -cocartesian square of spaces over X :

$$\begin{array}{ccc} Y & \xrightarrow{g} & X \\ g \downarrow & & \downarrow i_1 \\ X & \xrightarrow{i_0} & \mathcal{S}_X Y \end{array}$$

where i_0 and i_1 are the structure maps. By the Blakers-Massey Theorem, this square is $(2m+1)$ -cartesian. That is, the canonical map

$$Y \rightarrow \operatorname{holim}(X \xrightarrow{i_0} \mathcal{S}_X Y \xleftarrow{i_1} X) = \mathcal{O}_X \mathcal{S}_X Y$$

is $(2m+1)$ -connected. \square

Let

$$(\mathcal{A}') \quad \begin{array}{ccc} (A'_K, A'_L) & \longrightarrow & (W_K, W_L) \\ \downarrow & & \downarrow \\ (\mathcal{S}_X K, \mathcal{S}_X L) & \xrightarrow{\mathcal{S}_X f} & (X \times D^1, \partial(X \times D^1)) \end{array}$$

be a cofibrant and fibrant object of $E_{\mathcal{S}_X f}(\mathcal{S}_X K, X \times D^1 \operatorname{rel} \mathcal{S}_X L)$, which we also think of as the corresponding 0-cell of the space $\mathbb{E}_{\mathcal{S}_X f}(\mathcal{S}_X K, X \times D^1 \operatorname{rel} \mathcal{S}_X L)$. The underlying embedded thickening

$$\begin{array}{ccc} A'_L & \longrightarrow & W_L \\ \downarrow & & \downarrow \\ \mathcal{S}_X L & \longrightarrow & \partial(X \times D^1) \end{array}$$

is the stabilization of the embedded thickening with underlying map $L \rightarrow \partial X$ determined by the object \mathcal{D} above. Thus, we have an identification

$$(W_L, A'_L) \simeq (C_L \times D^1, \mathcal{S}_{C_L} A_L)$$

which allows us to write \mathcal{A}' as

$$\begin{array}{ccc}
(A'_K, \mathcal{S}_{C_L} A_L) & \longrightarrow & (W_K, C_L \times D^1) \\
\downarrow & & \downarrow \\
(\mathcal{S}_X K, \mathcal{S}_X L) & \xrightarrow{\mathcal{S}_X f} & (X \times D^1, \partial(X \times D^1))
\end{array}
\quad (\mathcal{A}')$$

Similarly, let

$$\begin{array}{ccc}
(A''_K, B''_L) & \longrightarrow & (W''_K, W''_L) \\
\downarrow & & \downarrow \\
(K, L) & \xrightarrow{f^1} & (X \times D^1, \partial(X \times D^1))
\end{array}
\quad (\mathcal{A}'')$$

be a cofibrant and fibrant object of $E_{f^1}(K, X \times D^1 \text{ rel } L)$, which we also think of as the corresponding 0-cell of $\mathbb{E}_{f^1}(K, X \times D^1 \text{ rel } L)$. According to [Kl02b, Proposition 4.1], there is a PD pair (C''_L, A''_L) and an identification

$$(W''_L, B''_L) \simeq (\mathcal{S}_X C''_L, \mathcal{S}_L A''_L)$$

along with an object $C''_K \in \mathcal{T}(C''_L \rightarrow X)$ which allow us to write \mathcal{A}'' as

$$\begin{array}{ccc}
(A''_K, \mathcal{S}_L A''_L) & \longrightarrow & (\mathcal{S}_X C''_K, \mathcal{S}_X C''_L) \\
\downarrow & & \downarrow \\
(K, L) & \xrightarrow{f^1} & (X \times D^1, \partial(X \times D^1))
\end{array}
\quad (\mathcal{A}'')$$

Remark 4.5. We will suppress the notation of pairs from \mathcal{A}' and \mathcal{A}'' , keeping in mind that we are working relative to the embedded thickenings of the underlying maps of subspaces that are already given (see Definition 2.3).

Applying the decomposition map to \mathcal{A}' , and applying the stabilization map to \mathcal{A}'' , we have

$$\begin{array}{ccc}
\mathcal{S}_{\mathcal{S}_X K} A'_K & \longrightarrow & \mathcal{S}_X W_K \\
\downarrow & & \downarrow \\
\mathcal{S}_X K & \longrightarrow & X \times D^2
\end{array}
\quad (\delta \mathcal{A}')
\qquad
\begin{array}{ccc}
\mathcal{S}_{\mathcal{S}_X C''_K} A''_K & \longrightarrow & \mathcal{S}_X C''_K \\
\downarrow & & \downarrow \\
\mathcal{S}_X K & \longrightarrow & X \times D^2
\end{array}
\quad (\sigma \mathcal{A}'')$$

where we have implicitly used the equivalence $D^1 \simeq *$ along with [Kl99, Lemma 2.5 (2)], to replace the spaces appearing in the lower left corner

of $\delta\mathcal{A}'$ and the upper right corners of both squares with homotopy equivalent spaces. Without loss of generality, assume that $\delta\mathcal{A}'$ and $\sigma\mathcal{A}''$ are both cofibrant and fibrant objects of the category $E_{\mathcal{S}_X f^1}(\mathcal{S}_X K, X \times D^2 \text{ rel } \mathcal{S}_X L)$. In the following lemmas, a *path* between objects of our categories will mean a *zig-zag of weak equivalences*.

Lemma 4.6. *Assume that there is a path from $\delta\mathcal{A}'$ to $\sigma\mathcal{A}''$ in the category*

$$E_{\mathcal{S}_X f^1}(\mathcal{S}_X K, X \times D^2 \text{ rel } \mathcal{S}_X L).$$

Then there is a weak equivalence

$$\mathcal{S}_X W_K \xrightarrow{\sim} \mathcal{S}_X C_K''$$

of spaces over X .

Proof. Without loss of generality, assume that W_K is cofibrant in $(\mathcal{T} \downarrow X)$. Since \mathcal{S}_X preserves cofibrant objects, the object $\mathcal{S}_X W_K$ is cofibrant. Furthermore, since \mathcal{A}'' is fibrant, we know that the object $\mathcal{S}_X C_K''$ is fibrant. Hence, the hypothesis of the lemma provides us with a path in $(\mathcal{T} \downarrow X)$ from a cofibrant object to a fibrant object. This induces an isomorphism in the homotopy category which then lifts back to the desired weak equivalence. \square

Remark 4.7. To show that the square $\mathcal{D}_{\sigma, \delta}$ is 0-cartesian, we will assume that $\delta\mathcal{A}'$ and $\sigma\mathcal{A}''$ lie in the same component of the space

$$\mathbb{E}_{\mathcal{S}_X f^1}(\mathcal{S}_X K, X \times D^2 \text{ rel } \mathcal{S}_X L)$$

and produce a 0-cell in $\mathbb{E}_f(K, X \text{ rel } L)$ (ie, an embedded thickening with underlying map f) that maps to both $\delta\mathcal{A}'$ and $\sigma\mathcal{A}''$, making the square $\mathcal{D}_{\sigma, \delta}$ commute up to preferred weak equivalence. In fact, it will be enough to assume, as above, that there is a path from $\delta\mathcal{A}'$ to $\sigma\mathcal{A}''$ in the category $E_{\mathcal{S}_X f^1}(\mathcal{S}_X K, X \times D^2 \text{ rel } \mathcal{S}_X L)$ since this will produce the desired path after passing to realizations.

Lemma 4.8. *Assume that there is a path from $\delta\mathcal{A}'$ to $\sigma\mathcal{A}''$ in the category*

$$E_{\mathcal{S}_X f^1}(\mathcal{S}_X K, X \times D^2 \text{ rel } \mathcal{S}_X L).$$

Then there are weak equivalences

- (i) $\mathcal{S}_{\mathcal{S}_X K} A'_K \xrightarrow{\sim} \mathcal{S}_{\mathcal{S}_X C_K''} A''_K$ in $(\mathcal{T} \downarrow \mathcal{S}_X K)$
- (ii) $\mathcal{S}_{\mathcal{S}_X C_K''} A''_K \xrightarrow{\sim} \mathcal{S}_{\mathcal{S}_X K} A'_K$ in $(\mathcal{T} \downarrow \mathcal{S}_X C_K'')$

Proof. Using the squares $\delta\mathcal{A}'$ and $\sigma\mathcal{A}''$ above, regard the spaces $\mathcal{S}_{\mathcal{S}_X K} A'_K$ and $\mathcal{S}_{\mathcal{S}_X C''_K} A''_K$ as objects of $(\mathcal{T} \downarrow \mathcal{S}_X K)$. Our assumption that \mathcal{A}' is cofibrant implies that A'_K is cofibrant, so that $\mathcal{S}_{\mathcal{S}_X K} A'_K$ is cofibrant. Our assumption that $\sigma\mathcal{A}''$ is fibrant implies that $\mathcal{S}_{\mathcal{S}_X C''_K} A''_K$ is a fibrant. Thus, as in the proof of the previous lemma, we have the weak equivalence (i) of spaces over $\mathcal{S}_X K$:

$$\mathcal{S}_{\mathcal{S}_X K} A'_K \xrightarrow{\sim} \mathcal{S}_{\mathcal{S}_X C''_K} A''_K.$$

Using the previous lemma to regard $\mathcal{S}_{\mathcal{S}_X K} A'$ as a space over $\mathcal{S}_{\mathcal{S}_X C''_K} A''$, a similar argument gives us the weak equivalence (ii) of spaces over $\mathcal{S}_X C''_K$. \square

Section Data. In the next two lemmas, we will use the weak equivalences above to construct sections to the maps $A'_K \rightarrow W_K$ and $A''_K \rightarrow K$ given the the diagrams \mathcal{A}' and \mathcal{A}'' . To avoid notational clutter, we will drop the subscript K throughout the rest of this section. Recall the square \mathcal{A}' (with the notation of pairs suppressed)

$$(\mathcal{A}') \quad \begin{array}{ccc} A' & \xrightarrow{\quad} & W \\ \downarrow & & \downarrow \\ \mathcal{S}_X K & \xrightarrow{\mathcal{S}_X f} & X \times D^1 \end{array}$$

Lemma 4.9. *Assume that $f: K \rightarrow X$ is r -connected ($r \geq 1$) and that $\text{hodim}(K, L) = k \leq n - 3$. Further, assume that $n \geq 2(k - r) + 2$. Then with the assumption of the previous lemma, there exists a map*

$$\mathcal{S}_W \emptyset = W \times S^0 \rightarrow A'$$

such that each of the restrictions $W \rightarrow A'$ is a section of the map $A' \rightarrow W$ in the diagram \mathcal{A}' .

Proof. Since f is r -connected, the natural map $\mathcal{S}_X K \rightarrow X$ is $(r + 1) \geq 2$ -connected. Hence, every local coefficient system on $\mathcal{S}_X K$ arises by pullback from one on X . An easy argument, using a relative Mayer-Vietoris sequence (with coefficients in any local system) shows that $\text{hodim}(\mathcal{S}_X K, \mathcal{S}_X L) \leq k + 1$. So, by definition, the map of spaces $A' \rightarrow \mathcal{S}_X K$ in \mathcal{A}' is $(n + 1) - (k + 1) - 1 = (n - k - 1)$ -connected. Furthermore, the map $\mathcal{S}_X f$ is $(r + 1)$ -connected. Since $k \leq n - 3$, we have that $n - k - 1 \geq 2$. Using [K199, Lemma 5.6 (2)], we infer that map $A' \rightarrow W$ in \mathcal{A}' is an $(r + 1)$ -connected map of spaces. Thus, the Blakers-Massey Theorem implies that the square

$$\begin{array}{ccc}
A' & \longrightarrow & W \\
\downarrow & & \downarrow \\
\mathcal{S}_X K & \longrightarrow & X \times D^1
\end{array}
\quad (\mathcal{A}')$$

$(n - k + r - 1)$ -cartesian. In particular, since \mathcal{A}' is fibrant, the map

$$A' \rightarrow \operatorname{holim}(\mathcal{S}_X K \rightarrow X \times D^1 \leftarrow W) = \mathcal{S}_X K \times_X W$$

is $(n - k + r - 1)$ -connected. Using this map and Lemma 4.4, form the square

$$\begin{array}{ccc}
A' & \longrightarrow & \mathcal{O}_{\mathcal{S}_X K} \mathcal{S}_{\mathcal{S}_X K} A' \\
\downarrow & & \downarrow \\
\mathcal{S}_X K \times_X W & \longrightarrow & \mathcal{O}_{\mathcal{S}_X K} \mathcal{S}_{\mathcal{S}_X K} (\mathcal{S}_X K \times_X W)
\end{array}
\quad (\mathcal{F})$$

of spaces over $\mathcal{S}_X K$. Let

$$F = \operatorname{hofiber}(A' \rightarrow \mathcal{S}_X K) \quad \text{and} \quad F' = \operatorname{hofiber}(\mathcal{S}_X K \times_X W \rightarrow \mathcal{S}_X K)$$

where the homotopy fibers are taken with respect to any basepoint in $\mathcal{S}_X K$. Then F and F' are both $(n - k - 2)$ -connected spaces, the latter being true since the connectivity of the space F' is equal to one less than the connectivity of the map $W \rightarrow X$. Furthermore, the map $F \xrightarrow{h} F'$ induced by the square \mathcal{F} is $(n - k + r - 1)$ -connected. The map h fits into the square obtained by taking homotopy fibers (over any basepoint of $\mathcal{S}_X K$) of the maps from \mathcal{F} . That is, we have a square

$$\begin{array}{ccc}
F & \longrightarrow & \Omega \Sigma F \\
\downarrow h & & \downarrow \\
F' & \longrightarrow & \Omega \Sigma F'
\end{array}
\quad (h\mathcal{F})$$

By Lemma 4.2, $h\mathcal{F}$ is $(2n - 2k + r - 3)$ -cartesian and, hence, so is \mathcal{F} . Using the evident map $\mathcal{S}_W \emptyset \rightarrow \mathcal{S}_X K$, as well as the weak equivalence (i) from Lemma 4.8, construct the composite

$$\mathcal{S}_{\mathcal{S}_X K}(\mathcal{S}_W \emptyset) \rightarrow \mathcal{S}_X(\mathcal{S}_W \emptyset) = \mathcal{S}_{\mathcal{S}_X W} \emptyset \rightarrow \mathcal{S}_{\mathcal{S}_X W} A'' \xrightarrow{\sim} \mathcal{S}_{\mathcal{S}_X K} A'$$

of spaces over $\mathcal{S}_X K$. Apply the functor $\mathcal{O}_{\mathcal{S}_X K}$ (restricted to $(\mathcal{T} \downarrow \mathcal{S}_X K)$) to get a map

$$\mathcal{O}_{\mathcal{S}_X K} \mathcal{S}_{\mathcal{S}_X K}(\mathcal{S}_W \emptyset) \rightarrow \mathcal{O}_{\mathcal{S}_X K} \mathcal{S}_{\mathcal{S}_X K} A'.$$

Precompose with the map from $\mathcal{S}_W \emptyset$ provided by Lemma 4.4 to get

$$\mathcal{S}_W \emptyset \rightarrow \mathcal{O}_{\mathcal{S}_X K} \mathcal{S}_{\mathcal{S}_X K} A'.$$

Finally, combine this map with the map

$$\mathcal{S}_W \emptyset = \mathcal{S}_X \emptyset \times_X W \rightarrow \mathcal{S}_X K \times_X W$$

and the square \mathcal{F} to form the following diagram of spaces over $\mathcal{S}_X K$:

$$\begin{array}{ccc} \mathcal{S}_W \emptyset & \xrightarrow{\quad} & \mathcal{O}_{\mathcal{S}_X K} \mathcal{S}_{\mathcal{S}_X K} A' \\ \downarrow \text{dashed} & \searrow & \downarrow \\ A' & \xrightarrow{\quad} & \mathcal{O}_{\mathcal{S}_X K} \mathcal{S}_{\mathcal{S}_X K} A' \\ \downarrow & & \downarrow \\ \mathcal{S}_X K \times_X W & \longrightarrow & \mathcal{O}_{\mathcal{S}_X K} \mathcal{S}_{\mathcal{S}_X K}(\mathcal{S}_X K \times_X W) \end{array}$$

By obstruction theory, the dashed arrow exists provided that $\text{hodim}(W) \leq 2n - 2k + r - 3$. Note that the codimension hypothesis $k \leq n - 3$ implies that $W \rightarrow X$ is 2-connected. Hence, using duality and excision, we have the following isomorphisms for all local coefficient systems:

$$H^*(W) \cong H_{n+1-*}(\overline{W}, A') \cong H_{n+1-*}(X \times D^1, \mathcal{S}_X K).$$

Since $(X \times D^1, \mathcal{S}_X K)$ is an $(r+1)$ -connected pair of spaces, the isomorphism above implies that W is cohomologically $(n-r-1)$ -dimensional (i.e., its cohomology vanishes in degrees $> n-r-1$). But $r \leq k \leq n-3$, so that $n-r-1 \geq 2$. Hence, by [GK08, Proposition 8.1], $\text{hodim}(W) \leq n-r-1$. Thus, the dashed arrow exists provided that $n-r-1 \leq 2n-2k+r-3$, which is equivalent to $n \geq 2(k-r)+2$. This establishes the existence of the map

$$\mathcal{S}_W \emptyset = W \times S^0 \rightarrow A'.$$

□

We now make a similar construction associated with the square \mathcal{A}'' :

Lemma 4.10. *With the assumptions of the previous lemma, there exists a map*

$$\mathcal{S}_K \emptyset = K \times S^0 \rightarrow A''$$

such that each of the restrictions $K \rightarrow A''$ is a section of the map $A'' \rightarrow K$ in the diagram \mathcal{A}'' .

Proof. By definition, the left vertical map in \mathcal{A}'' is $(n - k) \geq 3$ -connected, so the top horizontal map is r -connected by [KL99, Proposition 5.6 (2)]. By the Blakers-Massey Theorem, the square

$$\begin{array}{ccc} A'' & \longrightarrow & \mathcal{S}_X C \\ \downarrow & & \downarrow \\ K & \longrightarrow & X \times D^1 \end{array}$$

associated with \mathcal{A}'' is $(n - k + r - 1)$ -cartesian. Since \mathcal{A}'' was assumed fibrant, the map

$$A'' \rightarrow K \times_X \mathcal{S}_X C$$

is an $(n - k + r - 1)$ -connected map of spaces. Using Lemma 4.4, form the following commutative square of objects in $(\mathcal{T} \downarrow \mathcal{S}_X C)$:

$$(\mathcal{G}) \quad \begin{array}{ccc} A'' & \longrightarrow & \mathcal{O}_{\mathcal{S}_X C} \mathcal{S}_{\mathcal{S}_X C} A'' \\ \downarrow & & \downarrow \\ K \times_X \mathcal{S}_X C & \longrightarrow & \mathcal{O}_{\mathcal{S}_X C} \mathcal{S}_{\mathcal{S}_X C} (K \times_X \mathcal{S}_X C) \end{array}$$

Similar to the argument given in the previous lemma, this map gives rise to the square of homotopy fibers (for any choice of basepoint in $\mathcal{S}_X C$)

$$(\mathcal{H}' \mathcal{G}) \quad \begin{array}{ccc} G = \text{hofiber}(A'' \rightarrow \mathcal{S}_X C) & \longrightarrow & \Omega \Sigma G \\ \downarrow h' & & \downarrow \\ G' = \text{hofiber}(K \times_X \mathcal{S}_X C \rightarrow \mathcal{S}_X C) & \longrightarrow & \Omega \Sigma G' \end{array}$$

in which $\text{conn}(G) = \text{conn}(G') = r - 1$ and $\text{conn}(h') = (n - k + r - 1)$. By Lemma 4.2, $\mathcal{H}' \mathcal{G}$ is $(n - k + 2r - 2)$ -cartesian and, hence, so is \mathcal{G} . Using the evident map $\mathcal{S}_K \emptyset \rightarrow \mathcal{S}_X C$ and the weak equivalence (ii) from Lemma 4.8, construct the following composite in $(\mathcal{T} \downarrow \mathcal{S}_X C)$:

$$\mathcal{S}_{\mathcal{S}_X C}(\mathcal{S}_K \emptyset) \rightarrow \mathcal{S}_X(\mathcal{S}_K \emptyset) = \mathcal{S}_{\mathcal{S}_X K} \emptyset \rightarrow \mathcal{S}_{\mathcal{S}_X K} A' \xrightarrow{\sim} \mathcal{S}_{\mathcal{S}_X C} A''.$$

As above, apply the (restricted) functor $\mathcal{O}_{\mathcal{S}_X C}$ to get a map

$$\mathcal{O}_{S_X C} \mathcal{S}_{S_X C}(\mathcal{S}_K \emptyset) \rightarrow \mathcal{O}_{S_X C} \mathcal{S}_{S_X C} A''.$$

Precompose with the map from $\mathcal{S}_K \emptyset$ provided by Lemma 4.4 to get

$$\mathcal{S}_K \emptyset \rightarrow \mathcal{O}_{S_X C} \mathcal{S}_{S_X C} A''$$

and combine with the map $\mathcal{S}_K \emptyset = \mathcal{S}_X \emptyset \times_X K \rightarrow S_X C \times_X K$ and the square \mathcal{G} to form the diagram

$$\begin{array}{ccc} \mathcal{S}_K \emptyset & \xrightarrow{\quad} & \mathcal{O}_{S_X C} \mathcal{S}_{S_X C} A'' \\ \downarrow & \searrow \text{dashed} & \downarrow \\ K \times_X S_X C & \xrightarrow{\quad} & \mathcal{O}_{S_X C} \mathcal{S}_{S_X C}(K \times_X S_X C) \end{array}$$

Again, by obstruction theory, the dashed arrow exists provided that $k \leq n - k + 2r - 2$, which is equivalent to $n \geq 2(k - r) + 2$. Hence, we have the desired map

$$\mathcal{S}_K \emptyset = K \times S^0 \rightarrow A''.$$

□

5. PROOF OF THE STABILIZATION THEOREM

Recall the square $\mathcal{D}_{\sigma, \delta}$ from the previous section:

$$(\mathcal{D}_{\sigma, \delta}) \quad \begin{array}{ccc} \mathbb{E}_f(K, X \text{ rel } L) & \xrightarrow{\sigma} & \mathbb{E}_{S_X f}(\mathcal{S}_X K, X \times D^1 \text{ rel } \mathcal{S}_X L) \\ \downarrow & & \downarrow \\ \mathbb{E}_{f^1}(K, X \times D^1 \text{ rel } L) & \longrightarrow & \mathbb{E}_{S_X f^1}(\mathcal{S}_X K, X \times D^2 \text{ rel } \mathcal{S}_X L) \end{array}$$

Lemma 5.1. *Assume that f is r -connected ($r \geq 1$). Then the square $\mathcal{D}_{\sigma, \delta}$ is 0-cartesian provided that $\text{hodim}(K, L) = k \leq n - 3$ and $n \geq 2(k - r) + 3$.*

Proof. Using the projection $X \times D^1 \rightarrow X$, write the underlying squares of \mathcal{A}' and \mathcal{A}'' as follows:

$$\begin{array}{ccc}
A'_K & \longrightarrow & W_K \\
(\mathcal{A}') \downarrow & & \downarrow \\
\mathcal{S}_X K & \longrightarrow & X
\end{array}
\qquad
\begin{array}{ccc}
A''_K & \longrightarrow & \mathcal{S}_X C''_K \\
(\mathcal{A}'') \downarrow & & \downarrow \\
K & \longrightarrow & X
\end{array}$$

Using the fact that these squares are ∞ -cocartesian, along with the maps constructed in Lemmas 4.9 and 4.10, we have the following ‘excision’ weak equivalences

$$(i) \quad \overline{X} \cup_{W_K} \overline{A'_K} \xrightarrow{\sim} \overline{\mathcal{S}_X K} \cup_{A'_K} A'_K \xrightarrow{\sim} \mathcal{S}_X K$$

$$(ii) \quad \overline{X} \cup_K \overline{A''_K} \xrightarrow{\sim} X \cup_X \overline{\mathcal{S}_X C''_K} \xrightarrow{\sim} \mathcal{S}_X C''_K$$

and, using (i) (along with an argument similar to Lemma 4.6 for the first weak equivalence below)

$$\begin{aligned}
(iii) \quad \overline{\mathcal{S}_X C''_K} \cup_X \overline{\mathcal{S}_X K} &\xrightarrow{\sim} (\overline{X} \cup_{W_K} \overline{X}) \cup_X \overline{\mathcal{S}_X K} \\
&\xrightarrow{\sim} \overline{X} \cup_{W_K} \overline{\mathcal{S}_X K} \\
&\xrightarrow{\sim} (\overline{X} \cup_{W_K} \overline{A'_K}) \cup_{A'_K} \overline{\mathcal{S}_X K} \\
&\xrightarrow{\sim} \overline{\mathcal{S}_X K} \cup_{A'_K} \overline{\mathcal{S}_X K} \\
&\xrightarrow{\sim} \mathcal{S}_{\mathcal{S}_X K} A'_K
\end{aligned}$$

What we have so far can be combined to form the punctured 4-dimensional cubical diagram

$$\begin{array}{ccccc}
& & & W_K & \\
& & & \swarrow & \searrow \\
K & \longrightarrow & X & & \\
\downarrow & & \downarrow & & \downarrow \\
A''_K & \longrightarrow & \mathcal{S}_X C''_K & \longrightarrow & \mathcal{S}_{\mathcal{S}_X K} A'_K
\end{array}
\quad
\begin{array}{ccccc}
& & & W_K & \longrightarrow & A'_K \\
& & & \swarrow & \searrow & \swarrow \\
X & \longrightarrow & \mathcal{S}_X K & & & \\
\downarrow & & \downarrow & & \downarrow & \downarrow \\
\mathcal{S}_X C''_K & \longrightarrow & \mathcal{S}_{\mathcal{S}_X K} A'_K & & &
\end{array}$$

Let B denote the homotopy limit of this punctured 4-cube, so that we have an ∞ -cartesian 4-cube

$$\begin{array}{ccc}
B & \longrightarrow & W_K \\
\swarrow & & \swarrow \\
K & \longrightarrow & X \\
\downarrow & & \downarrow \\
A''_K & \longrightarrow & \mathcal{S}_X C''_K
\end{array}
\longrightarrow
\begin{array}{ccc}
W_K & \longrightarrow & A'_K \\
\swarrow & & \swarrow \\
X & \longrightarrow & \mathcal{S}_X K \\
\downarrow & & \downarrow \\
\mathcal{S}_X C''_K & \longrightarrow & \mathcal{S}_{\mathcal{S}_X K} A'_K
\end{array}$$

Remark 5.2. To avoid technical difficulties, we will assume that we have mapped the original punctured cube to a new punctured cube by a pointwise weak equivalence, and that the limit of the new punctured cube is the homotopy limit of the original punctured cube. The new punctured cube, together with its limit, is a strictly commutative cube. Hence, we will assume that the 4-cube above is strictly commutative, but will keep the notation as is.

We wish to apply the 4-D Face Theorem to this cube, so we check its hypotheses. As noted above, the cube is ∞ -cartesian. The weak equivalences (i), (ii), and (iii) above, along with the weak equivalences of Lemma 4.8 show that every 2-dimensional face which meets $\mathcal{S}_{\mathcal{S}_X K} A'_K$ is ∞ -cocartesian. So, by [Go92, Definition 2.5], every 3-dimensional face which meets $\mathcal{S}_{\mathcal{S}_X K} A'_K$ is strongly cocartesian. One can easily check that, in the notation of the 4-D Face Theorem, $k_1 = k_2 = n - k - 1$ and $k_3 = k_4 = r$. In particular, since $k \leq n - 3$, we have $k_1, k_2 \geq 2$. Hence, the theorem applies and as a consequence, we have that B is connected and that the square

$$(\mathcal{B}) \quad \begin{array}{ccc} B & \longrightarrow & K \\ \downarrow & & \downarrow \\ K & \longrightarrow & A''_K \end{array}$$

is $(2(n - k - 1) + 2r - 1) = (2n - 2k + 2r - 3)$ -cocartesian.

Claim. There exists a space A and a $(2n - 2k + 2r - 5)$ -connected map $A \rightarrow B$ such that the square

$$\begin{array}{ccc} A & \longrightarrow & K \\ \downarrow & & \downarrow \\ K & \longrightarrow & A''_K \end{array}$$

(with B replaced by A) is ∞ -cocartesian.

Proof of Claim. We wish to apply the *Cocartesian Replacement Theorem* [Kl99, Theorem 4.2], to the square \mathcal{B} above. Our hypotheses that $r \geq 1$ and $k \leq n - 3$ imply that $2(n - k + r) - 3 \geq 3$. Now, choose a basepoint for B (which, in turn, bases K and A''_K) and consider the map $K \vee K \rightarrow A''_K$. Since $A''_K \rightarrow K$ is $(n - k)$ -connected, we have a long exact sequence on cohomology (with respect to any local coefficient system on A''_K) given by

$$\cdots \rightarrow H^{*-1}(K \vee K) \rightarrow H^*(\overline{A''_K}, K \vee K) \rightarrow H^*(A''_K) \rightarrow \cdots$$

By definition, A''_K is a PD space of dimension n . So, since $k \leq n - 3$, the sequence above implies that the relative cohomology of $K \vee K \rightarrow A''_K$ vanishes in degrees $\geq n + 1$. That is, the relative cohomology of $K \vee K \rightarrow A''_K$ vanishes in degrees $> 2(n - k + r) - 3$ provided that $n + 1 \leq 2n - 2k + 2r - 2$, which is equivalent to $n \geq 2(k - r) + 3$. Hence, [Kl99, Theorem 4.2] gives the desired space A and proves the claim. \square

Now, consider one of the other 2-dimensional faces of the 4-cube labeled by

$$\begin{array}{ccc} B & \longrightarrow & W_K \\ \downarrow & & \downarrow \\ K & \longrightarrow & X \end{array}$$

Replacing B with the space A constructed in Claim, form the square

$$\begin{array}{ccc} A & \longrightarrow & W_K \\ \downarrow & & \downarrow \\ K & \longrightarrow & X \end{array}$$

By [Kl99, Claims 6.5 and 6.6], this square is an embedded thickening for f . Writing A_K for A and C_K for W_K , we have the desired object

$$\begin{array}{ccc} (A_K, A_L) & \longrightarrow & (C_K, C_L) \\ \downarrow & & \downarrow \\ (K, L) & \longrightarrow & (X, \partial X) \end{array}$$

of $E_f(K, X \text{ rel } L)$ and, hence, the desired 0-cell of $\mathbb{E}_f(K, X \text{ rel } L)$. This proves that $\mathcal{D}_{\sigma, \delta}$ is 0-cartesian. \square

Remark 5.3. To establish the connectivity of the stabilization map, we will use the identification

$$(*) \quad \pi_j(\mathbb{E}_f(K, X \text{ rel } L)) \cong \pi_0(\mathbb{E}_{f \times id}(K \times D^j, X \times D^j \text{ rel } \mathcal{S}_K^j L)).$$

To justify this identification, note that one can build the space of embedded thickenings for $f : K \rightarrow X$ as a semi-simplicial set ³ \mathcal{E}_f which satisfies the Kan condition. A j -simplex of \mathcal{E}_f is given by a square of $(j+2)$ -ads with lower horizontal map $K \times \Delta^j \rightarrow X \times \Delta^j$. Face maps are given by restriction. Applying the subdivision functor sd [GJ99, Page 183], to the nerve $N(E_f(K, X \text{ rel } L))$ (whose geometric realization is our space of embedded thickenings for f) gives an identification

$$\pi_0(\mathcal{E}_f) \cong \pi_0(sdN(E_f(K, X \text{ rel } L))).$$

One then has the identification $\pi_j(\mathcal{E}_f) = \pi_0(\Omega^j \mathcal{E}_f)$ (for some choice of basepoint vertex) where $\Omega^j \mathcal{E}_f$ is the iterated simplicial loop-space construction [GJ99, Section I.7]. Passing to geometric realizations then gives the identification $(*)$. We leave the details to the reader.

Recall the statement of Theorem A:

Theorem A (Stabilization). *Let $f : (K, L) \rightarrow (X, \partial X)$ be a map from a cofibration pair of homotopy finite spaces (K, L) , with $\text{hodim}(K, L) = k$, to a PD pair $(X, \partial X)$ of dimension n . Assume that $f : K \rightarrow X$ is r -connected ($r \geq 1$) and that $k \leq n - 3$. Then the stabilization map*

$$\sigma : \mathbb{E}_f(K, X \text{ rel } L) \rightarrow \mathbb{E}_{\mathcal{S}_X f}(\mathcal{S}_X K, X \times D^1 \text{ rel } \mathcal{S}_X L)$$

is $(n - 2(k - r) - 3)$ -connected.

Proof. Lemma 5.1 tells us that the square

$$\begin{array}{ccc} \mathbb{E}_f(K, X \text{ rel } L) & \xrightarrow{\sigma} & \mathbb{E}_{\mathcal{S}_X f}(\mathcal{S}_X K, X \times D^1 \text{ rel } \mathcal{S}_X L) \\ \downarrow & & \downarrow \\ \mathbb{E}_{f^1}(K, X \times D^1 \text{ rel } L) & \longrightarrow & \mathbb{E}_{\mathcal{S}_X f^1}(\mathcal{S}_X K, X \times D^2 \text{ rel } \mathcal{S}_X L) \end{array}$$

($\mathcal{D}_{\sigma, \delta}$)

is 0-cartesian, provided that $k \leq n - 3$ and $n \geq 2(k - r) + 3$. That is, under these assumptions, we have a 0-connected map

³Here, we mean “simplicial set without degeneracies”, ie Δ -set.

$$\mathbb{E}_f(K, X \text{ rel } L) \xrightarrow{\alpha} P$$

where P denotes the homotopy limit of the partial diagram gotten from $\mathcal{D}_{\sigma,\delta}$ by considering only the bottom horizontal and right vertical maps. Repeated application of the decompression map on both sides of $\mathcal{D}_{\sigma,\delta}$ forms an infinite tower of embedding spaces, and it is clear that one obtains contractible spaces on both sides of the tower after passing to homotopy colimits. That is, we have a weak equivalence after passing to colimits. Thus, by a downward induction on codimension, we may assume that the bottom horizontal map in $\mathcal{D}_{\sigma,\delta}$ is j -connected for some $j \geq 0$. This implies that the top horizontal map in the ∞ -cartesian square

$$\begin{array}{ccc} P & \xrightarrow{\beta} & \mathbb{E}_{\mathcal{S}_X f}(\mathcal{S}_X K, X \times D^1 \text{ rel } \mathcal{S}_X L) \\ \downarrow & & \downarrow \\ \mathbb{E}_{f^1}(K, X \times D^1 \text{ rel } L) & \longrightarrow & \mathbb{E}_{\mathcal{S}_X f^1}(\mathcal{S}_X K, X \times D^2 \text{ rel } \mathcal{S}_X L) \end{array}$$

is j -connected. Hence, we have a commutative diagram

$$\begin{array}{ccc} \mathbb{E}_f(K, X \text{ rel } L) & \xrightarrow{\sigma} & \mathbb{E}_{\mathcal{S}_X f}(\mathcal{S}_X K, X \times D^1 \text{ rel } \mathcal{S}_X L) \\ & \searrow \alpha & \nearrow \beta \\ & P & \end{array}$$

from which it is clear that σ is 0-connected. That is, the stabilization map induces a π_0 surjection provided that $k \leq n-3$ and $n \geq 2(k-r)+3$. Using Remark 5.3, we conclude that σ induces a surjection

$$\pi_j(\mathbb{E}_f(K, X \text{ rel } L)) \rightarrow \pi_j(\mathbb{E}_{\mathcal{S}_X f}(\mathcal{S}_X K, X \times D^1 \text{ rel } \mathcal{S}_X L))$$

provided that $j \leq n - 2(k - r) - 3$. This gives the desired surjectivity statement. For the injectivity statement, assume that we are given two PD embeddings \mathcal{D}_0 and \mathcal{D}_1 with underlying maps $f_0, f_1: K \rightarrow X$ and consider the associated embedding $\mathcal{D}_{K \amalg K}$ with underlying map $f_0 + f_1: K \amalg K \rightarrow \partial(X \times D^1)$. Further, assume that f_0 and f_1 give rise to the same embedding with underlying map $\mathcal{S}_X K \rightarrow X \times D^1$ after applying the stabilization map. Note that this embedding is relative to the embedding $\mathcal{D}_{K \amalg K}$. Then we have an associated map of pairs

$$F: (K \times D^1, K \amalg K) \rightarrow (X \times D^1, \partial(X \times D^1)).$$

Assume that $n \geq 2(k - r) + c$ for some constant c . This is equivalent to $r \geq 2k - n + (c - r)$. According to [KL02b, Corollary B], \mathcal{D}_0 is concordant to \mathcal{D}_1 provided that $c - r \leq 3$. But we have assumed that $r \geq 1$, so c is at least 4. Hence the induced map

$$\pi_j(\mathbb{E}_f(K, X \text{ rel } L)) \rightarrow \pi_j(\mathbb{E}_{\mathcal{S}_X f}(\mathcal{S}_X K, X \times D^1 \text{ rel } \mathcal{S}_X L))$$

is injective provided that $j \leq n - 2(k - r) - 4$. This completes the proof of Theorem A. \square

6. A GENERALIZATION OF SMOOTH STABILIZATION

We now prove the PD analog of the Stabilization Theorem of Conolly and Williams. To this end, fix an object $(f, K) \in (\mathcal{T} \downarrow S^n)$. Then there is a map

$$\mathbb{E}_{\mathcal{S}_n f}(\mathcal{S}_n K, S^n \times D^1 \text{ rel } \mathcal{S}_n \emptyset) \xrightarrow{c} \mathbb{E}_{\mathcal{S} f}(\mathcal{S} K, S^{n+1})$$

given by collapsing out the copies of S^n on either end of $S^n \times D^1$.

Lemma 6.1. *Assume that $\text{hodim}(K) = k \leq n - 3$. Then the “collapse” map c is 0-connected.*

Proof. Let

$$(\mathcal{D}_{\mathcal{S} f}) \quad \begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ \mathcal{S} K & \xrightarrow{\mathcal{S} f} & S^{n+1} \end{array}$$

denote a vertex of the space $\mathbb{E}_{\mathcal{S} f}(\mathcal{S} K, S^{n+1})$. To lift $\mathcal{D}_{\mathcal{S} f}$ back to a vertex of $\mathbb{E}_{\mathcal{S}_n f}(\mathcal{S}_n K, S^n \times D^1 \text{ rel } \mathcal{S}_n \emptyset)$, it will be enough to solve the lifting problem

$$\begin{array}{ccc} & \mathcal{S}_n K & \\ & \uparrow & \\ A & \xrightarrow{\quad} & \mathcal{S} K \\ & \downarrow q & \end{array}$$

Obstruction theory tells us that the problem has a solution provided that $\text{hodim}(A) \leq \text{conn}(q)$. Now, by definition A is a PD space of dimension n and, thus, is cohomologically n -dimensional. By [GK08, Proposition 8.1], we infer that $\text{hodim}(A) \leq n$ (This uses the assumption that $k \leq n - 3$, which implies that $n \geq 2$). Thus, it will be enough

to show that $\text{conn}(q) = n$. But this follows easily from the fact that $S^n \rightarrow *$ is n -connected.

□

Theorem B. *Let K be a homotopy finite complex with $\text{hodim}(K) = k \leq n - 3$. Assume that $f: K \rightarrow S^n$ is an r -connected map of spaces, $r \geq 1$. Then the induced map*

$$\pi_0(c \circ \sigma) : \pi_0(\mathbb{E}_f(K, S^n)) \rightarrow \pi_0(\mathbb{E}_{Sf}(\mathcal{S}K, S^{n+1}))$$

is surjective for $n \geq 2(k - r) + 3$ and injective for $n \geq 2(k - r) + 4$.

Proof. Invoke Theorem A and Lemma 6.1.

□

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